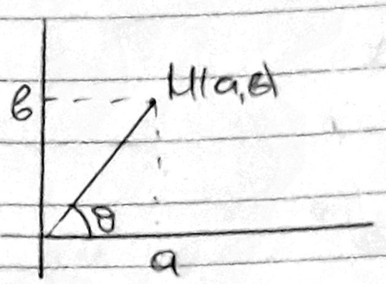
15/10/2018

Τριγωνομετρική μορφή μιγαδικών αριθμών



$z = a + bi$

$\rho = |z|$

$z = a + bi = \rho(\cos\theta + i\sin\theta)$

$$z_1 \cdot z_2 = \rho_1(\cos\theta_1 + i\sin\theta_1)\cdot \rho_2(\cos\theta_2 + i\sin\theta_2)$$
$$= \rho_1 \cdot \rho_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}\left(\cos(\theta_1-\theta_2) - i\sin(\theta_1-\theta_2)\right)$$

<u>Θεώρημα De Moivre</u> Αν $z = \rho(\cos\theta + i\sin\theta)$ και $n \in \mathbb{Z}$, τότε
$$z^n = \rho^n(\cos(n\theta) + i\sin(n\theta))$$

<u>Παράδειγμα</u>: Να λυθεί η εξίσωση $z^7 = 1$. Σίγουρα $z \neq 0$

Έστω $z = \rho(\cos\theta + i\sin\theta)$ λύση της $z^7 = 1$, τότε:

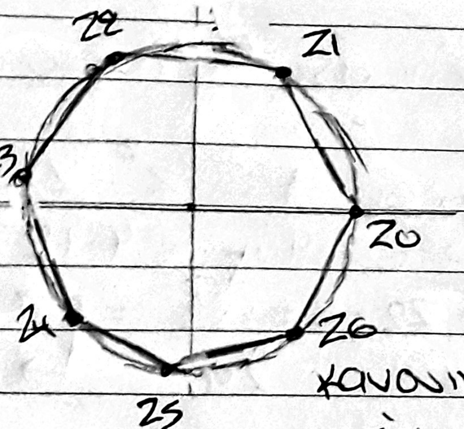
$$\rho^7\left(\cos(7\theta) + i\sin(7\theta)\right) = 1(\cos 0 + i\sin 0)$$

$$\Rightarrow \overset{\rho \geq 0}{\rho^7} = 1 \quad \text{και} \quad 7\theta - 0 = 2\kappa\pi \Rightarrow \theta = \frac{2\kappa\pi}{7},\ \kappa \in \mathbb{Z}$$

$\rho = 1$

<u>Λύσεις</u>:
- $z_0 = 1(\cos 0 + i\sin 0) = 1$
- $z_1 = \left(\cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)\right)$
- $z_2 = \left(\cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right)\right)$
- $z_3 = \left(\cos\left(\frac{6\pi}{7}\right) + i\sin\left(\frac{6\pi}{7}\right)\right)$

- $z_4 = \left(\cos\left(\frac{8\pi}{7}\right) + i\sin\left(\frac{8\pi}{7}\right)\right)$
- $z_5 = \left(\cos\left(\frac{10\pi}{7}\right) + i\sin\left(\frac{10\pi}{7}\right)\right)$
- $z_6 = \left(\cos\left(\frac{12\pi}{7}\right) + i\sin\left(\frac{12\pi}{7}\right)\right)$

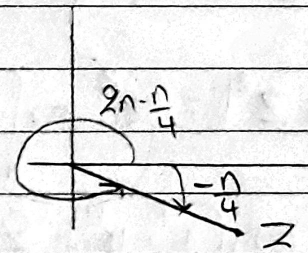


κανονικό επτάγωνο

Παράδειγμα: Να λυθεί η εξίσωση $z^4 = 16-16i$

Έστω z η λύση της εξίσωσης, τότε $z \neq 0$ και άρα $z = \rho(\cos\theta + i\sin\theta)$

$z^4 = 16-16i \Rightarrow \left[\rho(\cos\theta + i\sin\theta)\right]^4$



$\left[\rho(\cos\theta + i\sin\theta)\right]^4 = 16\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$|16-16i| = 16\sqrt{2}$

De Moivre:

$\rho^4\left(\cos(4\theta) + i\sin(4\theta)\right) = 16\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$\rho^4 = 16\sqrt{2}$ και $4\theta + \frac{\pi}{4} = 2\kappa\pi$

$\rho^4 = 16 \cdot 2^{1/2}$

$\rho = 2 \cdot 2^{1/8}$ και $4\theta = 2\kappa\pi - \frac{\pi}{4} \Rightarrow \theta = \frac{\kappa\pi}{2} - \frac{\pi}{16}$

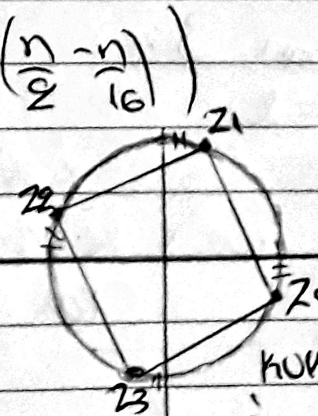
$\rho = 2\sqrt[8]{2}$

Λύσεις $\kappa=0$: $z_0 = 2\sqrt[8]{2}\left(\cos\left(-\frac{\pi}{16}\right) + i\sin\left(-\frac{\pi}{16}\right)\right)$

$\kappa=1$: $z_1 = 2\sqrt[8]{2}\left(\cos\left(\frac{\pi}{2} - \frac{\pi}{16}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{16}\right)\right)$

$\kappa=2$: $z_2 = 2\sqrt[8]{2}\left(\cos\left(\frac{4\pi}{2} - \frac{\pi}{16}\right) + i\sin\left(\frac{4\pi}{2} - \frac{\pi}{16}\right)\right)$

$\kappa=3$: $z_3 = 2\sqrt[8]{2}\left(\cos\left(\frac{3\pi}{2} - \frac{\pi}{16}\right) + i\sin\left(\frac{3\pi}{2} - \frac{\pi}{16}\right)\right)$



κύκλος ακτίνας $2\sqrt[8]{2}$

τετράγωνο

Ασκήσεις Φυλλάδιο #2

Άσκηση 1: $z^2 - 2z + 3 = 0$

$\Delta = \beta^2 - 4\alpha\gamma = 4 - 12 = -8$

$z_{1,2} = \dfrac{-\beta \pm i\sqrt{\Delta}}{2\alpha} = \begin{cases} 1 + \sqrt{2}\,i \\ 1 - \sqrt{2}\,i \end{cases}$

Άσκηση 2: $|z^2| = z^2$

$$z^2 = (x+yi)^2 = x^2 + y^2 i^2 + 2xyi = x^2 - y^2 + 2xyi$$

$$|z^2| = z^2 \Rightarrow \sqrt{(x^2-y^2)^2 + (2xy)^2} = x^2 - y^2 + 2xyi$$

$$\Rightarrow \sqrt{x^4 + y^4 - 2x^2y^2 + 4x^2y^2} = x^2 - y^2 + 2xyi$$

$$\Rightarrow \sqrt{(x^2+y^2)^2} = x^2 - y^2 + 2xyi$$

$$\Rightarrow x^2 + y^2 + 0i = x^2 - y^2 + 2xyi$$

$\Rightarrow x^2 + y^2 = x^2 - y^2$ και $2xy = 0$

$y^2 = -y^2$

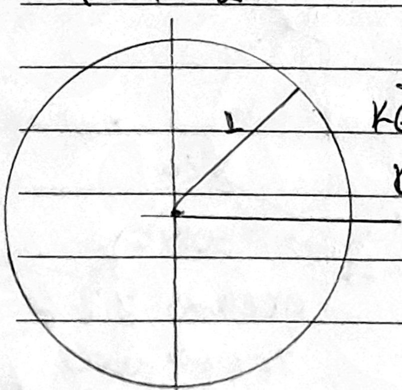
$2y^2 = 0$ $\qquad$ $xy = 0$

$y = 0$ $\qquad$ $x \cdot 0 = 0 \Rightarrow x \in R$

$\Rightarrow z = x + 0i$. Άρα το z είναι πραγματικός αριθμός.

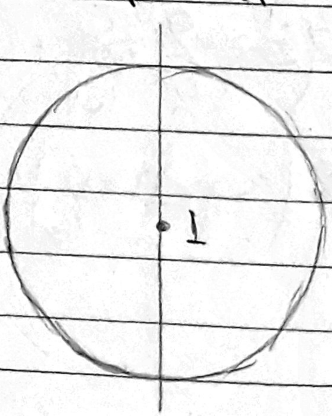
Άσκηση 3:

$|z| = 1$



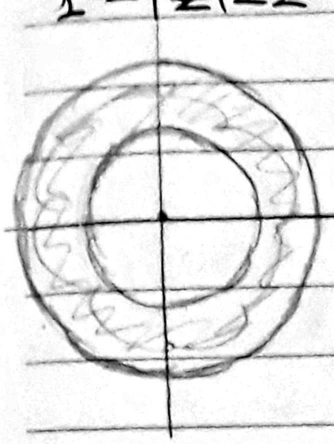
κύκλος με κέντρο (0,0) και ακτίνα 1

$|z - i| = 1$



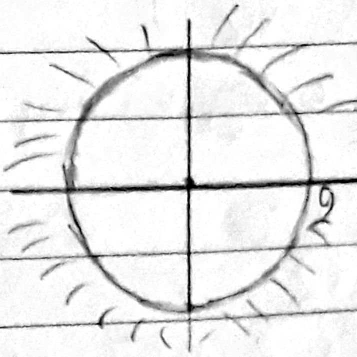
κύκλος με κέντρο (0,1) και ακτίνα 1

$1 < |z| < 2$



ανοικτός κυκλικός δίσκος.

$|z| \geq 2$



Το εξωτερικό του κυκλικού δίσκου με ακτίνα 2 και ο δίσκος του κύκλου.

Άσκηση 4: $A = \{ 2z+1 : z \in \mathbb{C}, |z| = 1 \}$

$|z| = 1$

$w = 2z+1$

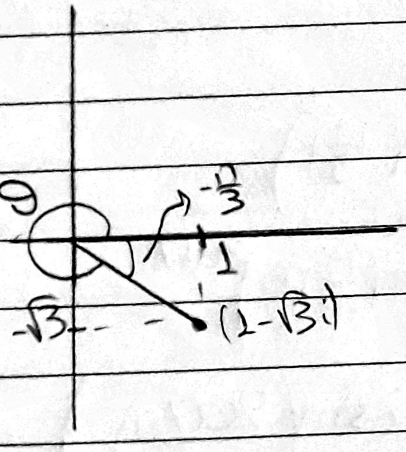
Έστω $w = 2z+1 \Rightarrow z = \frac{w-1}{2}$

$\left| \frac{w-1}{2} \right| = 1 \Rightarrow \frac{|w-1|}{|2|} = 1 \Rightarrow |w-1| = 2$ κύκλος με κέντρο $(1,0)$ και ακτίνα 1.

Άσκηση 5: • $1 - \sqrt{3}\,i$

$\rho = |1-\sqrt{3}i| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

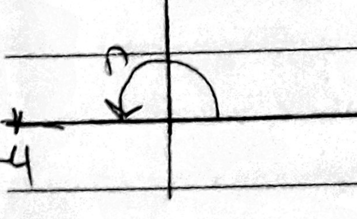
$1 - \sqrt{3}i = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$



$1 - \sqrt{3}i = 2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

• -4

$\rho = \sqrt{(-4)^2} = 4$

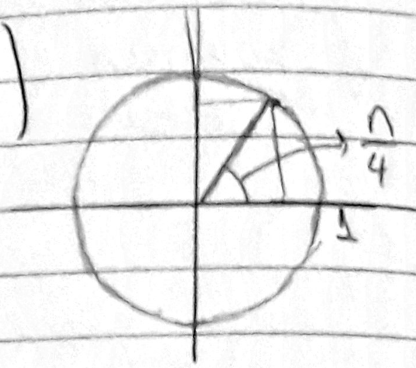


$-4 = 4\left[\cos(\pi) + i\sin(\pi)\right] =$

$= 4(\cos(\pi) + i\sin(\pi))$

Ασκηση 6: $\left(\frac{1+i}{\sqrt{2}}\right)^6$

$$\left(\frac{1+i}{\sqrt{2}}\right)^6 = \left(1\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right)$$



$$\frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\left|\frac{1+i}{\sqrt{2}}\right| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1} = 1$$

$$\frac{1+i}{\sqrt{2}} = 1\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

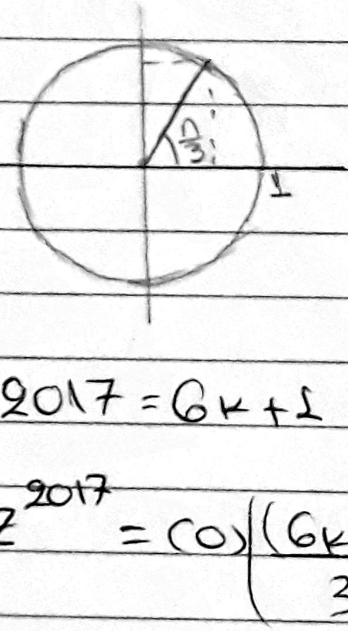
De Moivre:

$$1^6\left(\cos\left(\frac{6\pi}{4}\right) + i\sin\left(\frac{6\pi}{4}\right)\right) = 1\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right) = -i$$

Ασκηση 7: $z = \frac{1+i\sqrt{3}}{2}$ $\quad z^{2017}$?

$$z = \rho(\cos\theta + i\sin\theta)$$

$$z = \frac{1}{2} + \frac{i\sqrt{3}}{2} \qquad |z| = \frac{1}{4} + \frac{3}{4} = 1$$



$$z = 1\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$$

$$z^{2017} = \left(1\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)\right)^{2017}$$

$$= 1^{2017}\left(\cos\left(\frac{2017\pi}{3}\right) + i\sin\left(\frac{2017\pi}{3}\right)\right)$$

$$2017 = 6\kappa + 1$$

$$z^{2017} = \cos\left(\frac{(6\kappa+1)\pi}{3}\right) + i\sin\left(\frac{(6\kappa+1)\pi}{3}\right)$$

$$= \cos\left(2\kappa\pi + \frac{\pi}{3}\right) + i\sin\left(2\kappa\pi + \frac{\pi}{3}\right)$$

$$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1+i\sqrt{3}}{2}$$

Ασκηση 9: Να δειχθεί $z^n + \left(\frac{1}{z}\right)^n = 2\cos(n\theta)$

$z = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n} =$

$= \cos(n\theta) + i\sin(n\theta) + \cos(-n\theta) + i\sin(-n\theta) =$

$= \cos(n\theta) + \cos(-n\theta) = 2\cos(n\theta)$

$z \cdot \bar{z} = 1 \Rightarrow \frac{1}{z} = \bar{z}$